# DUAL -1 HAHN POLYNOMIALS: "CLASSICAL" POLYNOMIALS BEYOND THE LEONARD DUALITY

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Abstract. We introduce the -1 dual Hahn polynomials through an appropriate $q \to -1$ limit of the dual q-Hahn polynomials. These polynomials are orthogonal on a finite set of discrete points on the real axis, but in contrast to the classical orthogonal polynomials of the Askey scheme, the -1 dual Hahn polynomials do not exhibit the Leonard duality property. Instead, these polynomials satisfy a 4-th order difference eigenvalue equation and thus possess a bispectrality property. The corresponding generalized Leonard pair consists of two matrices $A, B$ each of size $N+1 \times N+1$. In the eigenbasis where the matrix $A$ is diagonal, the matrix $B$ is 3-diagonal; but in the eigenbasis where the matrix $B$ is diagonal, the matrix $A$ is 5-diagonal.

## 1. Introduction

Recently new explicit families of "classical" orthogonal polynomials $P_n(x)$ were introduced [12], [13], [14]. These polynomials satisfy an eigenvalue equation of the form

$$LP_n(x) = \lambda_n P_n(x). \tag{1.1}$$

The operator $L$ is of first order in the derivative operator $\partial_x$ and contains moreover the reflection operator $R$ defined by $Rf(x) = f(-x)$; it can be identified as a first order operator of Dunkl type written as

$$L = F(x)(I - R) + G(x)\partial_x R \tag{1.2}$$

with some real rational functions $F(x), G(x)$. The corresponding polynomial eigensolutions $P_n(x)$ can be obtained from the big and little q-Jacobi polynomials by an appropriate limit $q \to -1$.

In [11] we generalized this approach to the case of Dunkl shift operators. In this case the operator $L$ contains the shift operator $T^+f(x) = f(x+1)$ together with the reflection operator $R$:

$$L = F(x)(I - R) + G(x)(T^+R - I) \tag{1.3}$$

($I$ stands for the identity operator). The rational functions can be recovered from the condition that the operator $L$ stabilizes the spaces of polynomials, i.e. it sends any polynomial of degree $n$ into a polynomial of the same degree. It can then be demonstrated [11] that the polynomial eigensolutions $P_n(x)$ of the eigenvalue equation (1.1) with the operator (1.3) satisfy the 3-term recurrence relation and hence are orthogonal polynomials. In fact, the polynomials $P_n(x)$ in this case coincide with the Bannai-Ito (BI) polynomials first constructed in [1] (see also [10]).

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The BI polynomials thus possess the bispectrality property: they satisfy simultaneously the 3-term recurrence relation (common to all orthogonal polynomials)

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x) \tag{1.4}$$

and the eigenvalue equation (1.1). Moreover, in the case when the support of the orthogonality measure consists of a finite number of points $x_s,\ s = 0, 1, 2, \dots, N$, the BI polynomials satisfy the Leonard duality property [6], [1], [10]. This means that there is a finite difference equation of the form

$$U_s\left(P_n(x_{s+1}) - P_n(x_s)\right) + V_s\left(P_n(x_{s-1}) - P_n(x_s)\right) = \lambda_n P_n(x_s) \tag{1.5}$$

with some real coefficients $V_s, U_s$. The difference equation (1.5) is of the second order and can be considered as a dual relation with respect to the recurrence relation. In fact, the difference equation equation (1.5) is a simple consequence of the eigenvalue equation (1.1) with the Dunkl shift operator (1.3) [11].

We showed in [11] that the BI polynomials can be obtained by an appropriate $q \to -1$ limit from the Askey-Wilson polynomials. Correspondingly, the Dunkl shift operator $L$ appears in the same limit from the Askey-Wilson difference operator [11]. It should be stressed that there are several possibilities in taking the limit $q \to -1$ of the Askey-Wilson polynomials. Not all of them lead to the BI polynomials. There is another family of orthogonal polynomials - called the complementary BI polynomials in [11] - which can be obtained from the Askey-Wilson polynomials in the same limit. In contrast to the BI polynomials, the complementary BI (CBI) polynomials do not satisfy an eigenvalue equation of the form (1.1). As a consequence, the CBI polynomials do not possess the Leonard duality property (1.5).

The main purpose of the present paper is to study the orthogonal polynomials which appear in the $q \to -1$ limit of the dual q-Hahn polynomials. We call these polynomials the dual -1 Hahn polynomials. We derive explicitly basic properties and relations for them including a 3-term recurrence relation. The main result is the existence of a dual eigenvalue equation (1.1) for the dual -1 Hahn polynomials. In contrast to the BI polynomials, the operator $L$ is now 2-nd order with respect to the shift operators. As a consequence, the dual -1 Hahn polynomials obey a 5-term difference relation on an appropriate grid $x_s$ (instead of the 3-term relation (1.5) for the BI polynomials).

## 2. Dual q-Hahn polynomials

The dual $q$-Hahn polynomials [5]

$$R_n(x; a, b, N) = {}_3\Phi_2\left(\begin{matrix} q^{-n}, q^{-s}, abq^{s+1} \\ aq, q^{-N} \end{matrix} \mid q; q\right) \tag{2.1}$$

depend on 3 parameters $a, b, N$, where $N$ is a positive integer. The argument $x$ can be parametrized in terms of $s$ as follows

$$x = q^{-s} + abq^{s+1}. \tag{2.2}$$

The polynomials $R_n(x; a, b, N)$ satisfy the 3-term recurrence relation

$$A_n R_{n+1}(x; a, b, N) - (A_n + C_n - 1 - abq) R_n(x; a, b, N) + C_n R_{n-1}(x; a, b, N) = x R_n(x; a, b, N), \tag{2.3}$$

where

$$A_n = (1 - q^{n-N})(1 - aq^{n+1}), \quad C_n = aq(1 - q^n)(b - q^{n-N-1}). \tag{2.4}$$

The polynomials $R_n(x;a,b,N)$ are not monic; their monic version $P_n(x;a,b,N) = \kappa_n R_n(x;a,b,N) = x^n + O(x^{n-1})$ (with $\kappa_n$ an appropriate factor) will obey the recurrence relation

$$P_{n+1}(x;a,b,N) + b_n P_n(x;a,b,N) + u_n P_{n-1}(x;a,b,N) = xP_n(x;a,b,N), \tag{2.5}$$

where

$$b_n = 1 + abq - A_n - C_n, \quad u_n = A_{n-1}C_n.$$

The dual q-Hahn polynomials also verify a q-difference equation of second order [5]

$$B(s)R_n(x_{s+1}) + D(s)R_n(x_{s-1}) - (B(s)+D(s))R_n(x_s) = (q^{-n}-1)R_n(x_s), \quad s = 0,1,2,\dots,N, \tag{2.6}$$

where $x_s$ is given by (2.2) and

$$B(s) = \frac{(1-q^{s-N})(1-aq^{s+1})(1-abq^{s+1})}{(1-abq^{2s+1})(1-abq^{2s+2})}$$

$$D(s) = -\frac{aq^{s-N}(1-q^s)(1-abq^{s+1+N})(1-bq^s)}{(1-abq^{2s+1})(1-abq^{2s})}. \tag{2.7}$$

As per [7], the equation (2.6) can be interpreted as a q-difference equation on the "q-quadratic grid" $x_s$.

The dual q-Hahn polynomials satisfy the orthogonality relation

$$\sum_{s=0}^{N} w_s R_n(x_s) R_m(x_s) = \kappa_0 u_1 u_1 \dots u_n \delta_{nm}, \tag{2.8}$$

where the discrete weights are

$$w_s = \frac{(aq, abq, q^{-N}; q)_s}{(q, abq^{N+2}, bq; q)_s} \frac{1-abq^{2s+1}}{(1-abq)(-aq)^s}\, q^{Ns-s(s-1)/2} \tag{2.9}$$

and the normalization constant is

$$\kappa_0 = \frac{(abq^2;q)_N}{(bq;q)_N}(aq)^{-N}.$$

We used the standard notation for the q-shifted factorials [5].

When $q \to 1$ and $a = q^\alpha$, $b = q^\beta$ the dual q-Hahn polynomials become the ordinary dual Hahn polynomials [5]

$$W_n(x_s;a,b,N) = {}_3F_2\left(\begin{matrix} -n, -s, s+1+\alpha+\beta \\ \alpha+1, -N \end{matrix} \mid 1\right) \tag{2.10}$$

where

$$x_s = s(s+\alpha+\beta+1).$$

These polynomials satisfy the three-term recurrence relation

$$A_n W_{n+1}(x_s) - (A_n + C_n)W_n(x_s) + C_n W_{n-1}(x_s) = x_s W_n(x_s), \tag{2.11}$$

where

$$A_n = (n-N)(n+\alpha+1),\ C_n = n(n-\beta-N-1).$$

The corresponding monic dual Hahn polynomials $\hat{W}_n(x)$ obey

$$\hat{W}_{n+1}(x_s) + b_n \hat{W}_n(x_s) + u_n \hat{W}_{n-1}(x_s) = x_s \hat{W}_n(x_s), \tag{2.12}$$

where

$$u_n = A_{n-1}C_n = n(n-\beta-N-1)(n-N-1)(n+\alpha), \quad b_n = -A_n - C_n. \tag{2.13}$$

## 3. A limit $q \to -1$ of the recurrence relations and the orthogonality property

We wish to consider a limit $q \to -1$. We will assume that $a \to \pm 1$ and $b \to \pm 1$ when $q \to -1$. We want to obtain a nondegenerate limit of the coefficients $A_n(1+q)^{-1}, C_n(1+q)^{-1}$ for $q \to -1$. This means that both these limit coefficients should exist and be nonzero for all admissible values $n = 1, 2, \dots, N$. It is hence easily seen that necessarily, we must have $ab \to 1$. Two situations have to be considered separately:

(i) when $N = 2, 4, 6, \dots$ is even, the nontrivial $q \to -1$ limit then exists iff $a \to 1,\ b \to 1$. It is then natural to take the parametrization

$$q = -e^{\varepsilon},\ a = e^{-\alpha\varepsilon},\ b = e^{-\beta\varepsilon}, \quad \varepsilon \to 0 \tag{3.1}$$

with real parameters $\alpha, \beta$. Dividing the recurrence relation (2.3) by $q+1$ and taking the limit $\varepsilon \to 0$ we obtain the recurrence relation

$$\begin{aligned} A_n^{(-1)} R_{n+1}^{(-1)}(y_s;\alpha,\beta,N) + C_n^{(-1)} R_{n-1}^{(-1)}(y_s;\alpha,\beta,N) - \\ (A_n^{(-1)} + C_n^{(-1)}) R_n^{(-1)}(y_s;\alpha,\beta,N)) = y_s R_n^{(-1)}(y_s;\alpha,\beta,N), \end{aligned} \tag{3.2}$$

where the grid $y_s$ has the following expression

$$y_s = \begin{cases} -\alpha - \beta + 2s + 1 & \text{if} \quad s \quad \text{even}, \\ \alpha + \beta - 2s - 1 & \text{if} \quad s \quad \text{odd} \end{cases} \tag{3.3}$$

or, equivalently,

$$y_s = (-1)^s\,(1 - \alpha - \beta + 2s), \quad s = 0, 1, \dots, N. \tag{3.4}$$

The recurrence coefficients are

$$A_n^{(-1)} = \begin{cases} 2(n-N) & \text{if} \quad n \quad \text{even} \\ 2(n+1-\alpha) & \text{if} \quad n \quad \text{odd} \end{cases}, \quad C_n^{(-1)} = \begin{cases} -2n & \text{if} \quad n \quad \text{even} \\ 2(N+1-\beta-n) & \text{if} \quad n \quad \text{odd} \end{cases}. \tag{3.5}$$

The corresponding monic $-1$ dual Hahn polynomials satisfy relation (2.5), where

$$u_n^{(-1)} = A_{n-1}^{(-1)} C_n^{(-1)} = \begin{cases} 4n(\alpha - n) & \text{if} \quad n \quad \text{even} \\ 4(N-n+1)(n+\beta-N-1) & \text{if} \quad n \quad \text{odd} \end{cases} \tag{3.6}$$

and

$$b_n^{(-1)} = 1 - \alpha - \beta - A_n^{(-1)} - C_n^{(-1)} = \begin{cases} 2N + 1 - \alpha - \beta & \text{if} \quad n \quad \text{even} \\ -2N - 3 + \alpha + \beta & \text{if} \quad n \quad \text{odd} \end{cases} \tag{3.7}$$

It is convenient to introduce the "$\mu$-numbers"

$$[n]_\mu = n + \mu(1 - (-1)^n), \tag{3.8}$$

which appear naturally in problems connected with the Dunkl operators [8]. One can then present the recurrence coefficients in the compact form

$$u_n^{(-1)} = 4[n]_\xi [N-n+1]_\eta, \quad b_n^{(-1)} = 2([n]_\xi + [N-n]_\eta) + 1 - \alpha - \beta, \tag{3.9}$$

where

$$\xi = \frac{\beta - N - 1}{2},\ \eta = \frac{\alpha - N - 1}{2}.$$

It is seen that $u_0 = u_{N+1} = 0$ as required for finite orthogonal polynomials. The positivity condition $u_n > 0,\ n = 1, 2, \dots, N$ is equivalent to the conditions

$$\alpha > N,\ \beta > N. \tag{3.10}$$

These polynomials are orthogonal on the finite set of points $y_s$

$$\sum_{s=0}^{N} w_s R_n^{(-1)}(y_s) R_m^{(-1)}(y_s) = \kappa_0 u_1^{(-1)} u_2^{(-1)} \dots u_n^{(-1)} \, \delta_{nm}, \tag{3.11}$$

where the discrete weights are defined as

$$w_{2s} = (-1)^s \frac{(-N/2)_s}{s!} \frac{(1-\alpha/2)_s (1-\alpha/2-\beta/2)_s}{(1-\beta/2)_s (N/2+1-\alpha/2-\beta/2)_s}, \quad s = 0, 1, 2, \dots, \frac{N}{2} \tag{3.12}$$

and

$$w_{2s+1} = (-1)^s \frac{(-N/2)_{s+1}}{s!} \frac{(1-\alpha/2)_s (1-\alpha/2-\beta/2)_s}{(1-\beta/2)_s (N/2+1-\alpha/2-\beta/2)_{s+1}}, \quad s = 0, 1, \dots, \frac{N}{2} - 1. \tag{3.13}$$

The normalization coefficient is

$$\kappa_0 = \frac{\left(1 - \frac{\alpha+\beta}{2}\right)_{N/2}}{\left(1 - \frac{\beta}{2}\right)_{N/2}}. \tag{3.14}$$

Assume that $\alpha = N + \epsilon_1, \ \beta = N + \epsilon_2$, where $\epsilon_{1,2}$ are arbitrary positive parameters. This parametrization corresponds to the positive condition for the dual -1 Hahn polynomials. Then it is easily verified that all the weights are positive $w_s > 0, \ s = 0, 1, \dots, N$.

Moreover, the spectral points $y_s$ are divided into two non-overlapping discrete sets of the real line:

$$\{1 - \delta, -3 - \delta, -7 - \delta, \dots, -2N + 1 - \delta\}$$

and

$$\{1 + \delta, 5 + \delta, 9 + \delta, \dots, 2N - 3 + \delta\},$$

where $\delta = \epsilon_1 + \epsilon_2 > 0$. The first set corresponds to $y_s$ with even $s$ and contains $1 + N/2$ points; the second set corresponds to $y_s$ with odd $s$ and contains $N/2$ points.

(ii) when $N = 1, 3, 5, \dots$ is odd, a nontrivial $q \to -1$ limit also exists iff $a \to -1, \ b \to -1$. We take the parametrization

$$q = -e^{\varepsilon}, \ a = -e^{\alpha\varepsilon}, \ b = -e^{\beta\varepsilon}, \quad \varepsilon \to 0 \tag{3.15}$$

with real parameters $\alpha, \beta$. Dividing again the recurrence relation (2.3) by $q+1$ and taking the limit $\varepsilon \to 0$, we obtain the recurrence relation (3.2) where the grid $y_s$ is defined as

$$y_s = \begin{cases} \alpha + \beta + 2s + 1 & \text{if} \quad s \quad \text{even} \\ -\alpha - \beta - 2s - 1 & \text{if} \quad s \quad \text{odd} \end{cases} \tag{3.16}$$

and the recurrence coefficients given by

$$A_n^{(-1)} n = \begin{cases} 2(\alpha + n + 1) & \text{if} \quad n \quad \text{even} \\ 2(n - N) & \text{if} \quad n \quad \text{odd} \end{cases}, \quad C_n^{(-1)} = \begin{cases} -2n & \text{if} \quad n \quad \text{even} \\ 2(\beta + N - n + 1) & \text{if} \quad n \quad \text{odd} \end{cases}. \tag{3.17}$$

The corresponding monic $-1$ dual Hahn polynomials satisfy the standard relation (2.5), where

$$u_n^{(-1)} = A_{n-1}^{(-1)} C_n^{(-1)} = \begin{cases} 4n(N + 1 - n) & \text{if} \quad n \quad \text{even} \\ 4(\alpha + n)(\beta + N + 1 - n) & \text{if} \quad n \quad \text{odd} \end{cases} \tag{3.18}$$

and

$$b_n^{(-1)} = 1+\alpha+\beta - A_n^{(-1)} - C_n^{(-1)} = \begin{cases} -1-\alpha+\beta & \text{if} \quad n \quad \text{even} \\ -1+\alpha-\beta & \text{if} \quad n \quad \text{odd} \end{cases}. \tag{3.19}$$

Again, as in the case of even $N$, it is possible to present the recurrence coefficients in the compact form

$$u_n^{(-1)} = 4[n]_\xi[N-n+1]_\eta, \quad b_n^{(-1)} = 2([n]_\xi + [N-n]_\eta) - 2N - 1 - \alpha - \beta, \tag{3.20}$$

with $\xi = \alpha/2$, $\eta = \beta/2$. It is seen that in both cases: $N$ even and $N$ odd, the recurrence coefficients of the dual -1 Hahn polynomials are presented in the unified form (3.9) or (3.20) with the difference only residing with the parameters $\xi, \eta$.

It is seen that $u_0 = u_{N+1} = 0$ as required for finite orthogonal polynomials. The positivity condition $u_n > 0$, $n = 1, 2, \dots, N$ is equivalent either to condition

$$\alpha > -1, \ \beta > -1 \tag{3.21}$$

or to condition $\alpha < -N$, $\beta < -N$. In what follows we shall use only condition (3.21).

The polynomials $R_n^{(-1)}(x)$ are orthogonal on the finite set of points $y_s$

$$\sum_{s=0}^{N} w_s R_n^{(-1)}(y_s) R_m^{(-1)}(y_s) = \kappa_0 u_1^{(-1)} u_2^{(-1)} \dots u_n^{(-1)} \, \delta_{nm}, \tag{3.22}$$

where the discrete weights are defined as
(3.23)

$$w_{2s} = (-1)^s \frac{(-(N-1)/2)_s}{s!} \frac{(1/2+\alpha/2)_s(1+\alpha/2+\beta/2)_s}{(1/2+\beta/2)_s(N/2+3/2+\alpha/2+\beta/2)_s}, \quad s = 0, 1, 2, \dots, \frac{N-1}{2}$$

and
(3.24)

$$w_{2s+1} = (-1)^s \frac{(-(N-1)/2)_s}{s!} \frac{(1/2+\alpha/2)_{s+1}(1+\alpha/2+\beta/2)_s}{(1/2+\beta/2)_{s+1}(N/2+3/2+\alpha/2+\beta/2)_s}, \quad s = 0, 1, 2, \dots, \frac{N-1}{2}$$

The normalization coefficient is

$$\kappa_0 = \frac{\left(1 + \frac{\alpha+\beta}{2}\right)_{(N+1)/2}}{\left(\frac{\beta+1}{2}\right)_{(N+1)/2}}. \tag{3.25}$$

Assume that $\alpha = -1 + \epsilon_1$, $\beta = -1 + \epsilon_2$, where $\epsilon_{1,2}$ are arbitrary positive parameters. This parametrization corresponds to the positive condition for the dual -1 Hahn polynomials for $N$ odd. Then it is easily verified that the weights are positive $w_s > 0$, $s = 0, 1, \dots, N$.

Moreover, the spectral points $y_s$ are divided into two non-overlapped discrete sets of the real line:

$$\{-1-\delta, -5-\delta, -9-\delta, \dots, -2N+1-\delta\}$$

and

$$\{-1+\delta, 3+\delta, 7+\delta, \dots, 2N-3+\delta\},$$

where $\delta = \epsilon_1 + \epsilon_2 > 0$. Both sets contain $(N-1)/2$ points.

## 4. Explicit expression in terms of the ordinary dual Hahn polynomials

In this section, we derive an explicit expression for the -1 dual Hahn polynomials in terms of the ordinary dual Hahn polynomials.

Consider first the case of even $N = 2, 4, 6, \dots$. Introduce the "shifted" monic -1 dual Hahn polynomials $\tilde{R}_n(x) = R_n^{(-1)}(x-1)$ From formulas (3.6), (3.7), we conclude that these polynomials satisfy the recurrence relation

$$\tilde{R}_{n+1}(x) + (-1)^n \tau \tilde{R}_n(x) + u_n \tilde{R}_{n-1}(x) = x\tilde{R}_n(x), \tag{4.1}$$

where

$$\tau = 2N + 2 - \alpha - \beta$$

and $u_n$ are given by (3.6).

A recurrence relation of the type (4.1) leads to orthogonal polynomials $\tilde{R}_n(x)$ which are very close to symmetric orthogonal polynomials. Using methods developed in [2] and [13], we can introduce a pair of monic orthogonal polynomials $P_n(x)$ and $Q_n(x)$ by the formulas :

$$\tilde{R}_{2n}(x) = P_n(x^2), \quad \tilde{R}_{2n+1}(x) = (x-\tau)Q_n(x^2). \tag{4.2}$$

It can easily be shown that the polynomials $P_n(x)$ and $Q_n(x)$ satisfy the following recurrence relations (it is assumed that $u_0 = 0$)

$$P_{n+1}(x) + (u_{2n} + u_{2n+1} + \tau^2)P_n(x) + u_{2n}u_{2n-1}P_{n-1}(x) = xP_n(x) \tag{4.3}$$

and

$$Q_{n+1}(x) + (u_{2n+2} + u_{2n+1} + \tau^2)Q_n(x) + u_{2n}u_{2n+1}Q_{n-1}(x) = xQ_n(x), \tag{4.4}$$

and moreover that the polynomials are connected by the Christoffel transform

$$Q_n(x) = \frac{P_{n+1}(x) + u_{2n+1}P_n(x)}{x - \tau^2}. \tag{4.5}$$

It is also easily seen that both $P_n(x)$ and $Q_n(x)$ are ordinary dual Hahn polynomials. We hence have the following explicit expression:

$$R_{2n}^{(-1)}(x-1) = \gamma_n^{(0)}\, {}_3F_2\left( {-n, \eta + \frac{x}{4}, \eta - \frac{x}{4} \atop -\frac{N}{2}, 1 - \frac{\alpha}{2}} \mid 1 \right), \quad n = 0, 1, 2, \dots \tag{4.6}$$

and

$$R_{2n+1}^{(-1)}(x-1) = \gamma_n^{(1)}\,(x-\tau)\, {}_3F_2\left( {-n, \eta + \frac{x}{4}, \eta - \frac{x}{4} \atop 1 - \frac{N}{2}, 1 - \frac{\alpha}{2}} \mid 1 \right), \quad n = 0, 1, 2, \dots, \tag{4.7}$$

where $\eta = 1/2 - (\alpha+\beta)/4$ and the normalization coefficients are

$$\gamma_n^{(0)} = 16^n(-N/2)_n(1-\alpha/2)_n, \quad \gamma_n^{(1)} = 16^n(1-N/2)_n(1-\alpha/2)_n.$$

Quite similarly, for the odd $N = 1, 3, 5, \dots$ we find

$$R_{2n}^{(-1)}(x-1) = \gamma_n^{(0)}\, {}_3F_2\left( {-n, \eta + \frac{x}{4}, \eta - \frac{x}{4} \atop -\frac{N-1}{2}, \frac{\alpha+1}{2}} \mid 1 \right), \quad n = 0, 1, 2, \dots \tag{4.8}$$

and

$$R_{2n}^{(-1)}(x-1) = \gamma_n^{(1)}(x+\alpha-\beta)\, {}_3F_2\left( {-n, \eta + \frac{x}{4}, \eta - \frac{x}{4} \atop -\frac{N-1}{2}, \frac{\alpha+3}{2}} \mid 1 \right), \quad n = 0, 1, 2, \dots, \tag{4.9}$$

where

$$\eta = \frac{\alpha+\beta+2}{4}, \ \gamma_n^{(0)} = 16^n \left(\frac{1-N}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n, \ \gamma_n^{(1)} = 16^n \left(\frac{1-N}{2}\right)_n \left(\frac{\alpha+3}{2}\right)_n.$$

Some of these polynomials have appeared in [9], [4] in the context of quantum spin chains.

## 5. Difference equation

Consider the following operator $L$ defined on the space of functions $f(s)$ that depend on a discrete variable $s$:

$$Lf(s) = B(s)f(s+1) + D(s)f(s-1) - (B(s)+D(s))f(s), \quad s = 0,1,2,\dots, \tag{5.1}$$

where $B(s), D(s)$ are given in (2.7). Manifestly, the difference equation (2.6) means that the dual $q$-Hahn polynomials are eigenfunctions of the operator $L$

$$LR_n(x(s)) = \lambda_n R_n(x(s)) \tag{5.2}$$

with the eigenvalues

$$\lambda_n = q^{-n} - 1. \tag{5.3}$$

When $q \to 1$ we obtain the difference eigenvalue equation for the ordinary dual Hahn polynomials

$$L_1 W_n(x(s)) = -nW_n(x(s)), \tag{5.4}$$

where the operator $L_1$ can be obtained from $L$ as $L_1 = \lim_{q\to 1} L(q-1)^{-1}$. Explicitly [5]

$$L_1 = B_1(s)f(s+1) + D_1(s)f(s-1) - (B_1(s)+D_1(s))f(s), \tag{5.5}$$

where

$$B_1(s) = \frac{(s+\alpha+\beta+1)(s+\alpha+1)(N-s)}{(2s+\alpha+\beta+1)(2s+\alpha+\beta+2)}, \quad D_1(s) = \frac{s(s+\beta)(s+\alpha+\beta+N+1)}{(2s+\alpha+\beta+1)(2s+\alpha+\beta)}.$$

When we try to perform a similar procedure for the limit $q \to -1$ we encounter a problem. Indeed, it is easily seen that $L(1+q)^{-1}$ does not have a nondegenerate limit as $q \to -1$. We thus cannot obtain an eigenvalue equation in 3-diagonal form like (5.4) for the dual -1 Hahn polynomials.

Nevertheless we observe that the operator $(L^2+2L)(1+q)^{-1}$ does survive in the limit $q \to -1$. We hence have the following eigenvalue equation for the dual -1 Hahn polynomials

$$HR_n^{(-1)}(y_s) = 2nR_n^{(-1)}(y_s), \tag{5.6}$$

where the grid $y_s$ is defined by (3.3) or (3.16) and

$$H = \lim_{q\to -1}(L^2+2L)(1+q)^{-1}.$$

The operator $H$ obviously is 5-diagonal, i.e.

(5.7)

$$Hf(s) = U_2(s)(f(s+2)-f(s)) + U_1(s)(f(s+1)-f(s)) + V_2(s)(f(s-2)-f(s)) + V_1(s)(f(s-1)-f(s))$$

The explicit expressions for the coefficients $U_i(s), V_i(s)$ depend on the parity of $N$. For even $N = 2, 4, 6, \dots$ they are:

$$U_2(s) = \begin{cases} -2\,\frac{(\alpha-s-2)(\beta+\alpha-s-2)(N-s)}{(\alpha+\beta-2\,s-2)(\alpha+\beta-2\,s-4)} & \text{if} \quad s \quad \text{even} \\ -2\,\frac{(\alpha-s-1)(\beta+\alpha-s-1)(N-s-1)}{(\alpha+\beta-2\,s-2)(\alpha+\beta-2\,s-4)} & \text{if} \quad s \quad \text{odd} \end{cases}, \tag{5.8}$$

$$U_1(s) = \begin{cases} 2\,\frac{(\beta+\alpha)(\alpha-\beta)(N-s)}{(\alpha+\beta-2\,s)(\alpha+\beta-2\,s-2)(\alpha+\beta-2\,s-4)} & \text{if} \quad s \quad \text{even} \\ 4\,\frac{(\alpha-s-1)(\beta+\alpha-s-1)(2\,N+2-\alpha-\beta)}{(\alpha+\beta-2\,s)(\alpha+\beta-2\,s-2)(\alpha+\beta-2\,s-4)} & \text{if} \quad s \quad \text{odd} \end{cases}, \tag{5.9}$$

$$V_2(s) = \begin{cases} 2\,\frac{s(\beta-s)(-\alpha-\beta+N+s)}{(\alpha+\beta-2\,s+2)(\alpha+\beta-2\,s)} & \text{if} \quad s \quad \text{even} \\ 2\,\frac{(s-1)(\beta-s+1)(-\alpha-\beta+N+s+1)}{(\alpha+\beta-2\,s+2)(\alpha+\beta-2\,s)} & \text{if} \quad s \quad \text{odd} \end{cases}, \tag{5.10}$$

$$V_1(s) = \begin{cases} 4\,\frac{s(\beta-s)(2\,N+2-\alpha-\beta)}{(\alpha+\beta-2\,s)(\alpha+\beta-2\,s-2)(\alpha+\beta-2\,s+2)} & \text{if} \quad s \quad \text{even} \\ -2\,\frac{(\beta+\alpha)(\alpha-\beta)(-\alpha-\beta+N+s+1)}{(\alpha+\beta-2\,s)(\alpha+\beta-2\,s-2)(\alpha+\beta-2\,s+2)} & \text{if} \quad s \quad \text{odd} \end{cases}; \tag{5.11}$$

and for odd $N = 1, 3, 5, \dots$

$$U_2(s) = \begin{cases} -2\,\frac{(\alpha+\beta+s+2)(\alpha+s+1)(N-s-1)}{(\alpha+\beta+2\,s+2)(\alpha+\beta+2\,s+4)} & \text{if} \quad s \quad \text{even} \\ -2\,\frac{(\alpha+s+2)(\alpha+\beta+s+1)(N-s)}{(\alpha+\beta+2\,s+2)(\alpha+\beta+2\,s+4)} & \text{if} \quad s \quad \text{odd} \end{cases}, \tag{5.12}$$

$$U_1(s) = \begin{cases} -2\,\frac{(\alpha+\beta)(\alpha+s+1)(\alpha+\beta+2\,N+2)}{(\alpha+\beta+2\,s)(\alpha+\beta+2\,s+2)(\alpha+\beta+2\,s+4)} & \text{if} \quad s \quad \text{even} \\ -4\,\frac{(\alpha-\beta)(N-s)(\alpha+\beta+s+1)}{(\alpha+\beta+2\,s)(\alpha+\beta+2\,s+2)(\alpha+\beta+2\,s+4)} & \text{if} \quad s \quad \text{odd} \end{cases}, \tag{5.13}$$

$$V_2(s) = \begin{cases} -2\,\frac{s(\beta+s-1)(\alpha+\beta+N+s+1)}{(\alpha+\beta+2\,s-2)(\alpha+\beta+2\,s)} & \text{if} \quad s \quad \text{even} \\ -2\,\frac{(s-1)(\beta+s)(\alpha+\beta+N+s)}{(\alpha+\beta+2\,s-2)(\alpha+\beta+2\,s)} & \text{if} \quad s \quad \text{odd} \end{cases}, \tag{5.14}$$

$$V_1(s) = \begin{cases} -4\,\frac{s(\alpha-\beta)(\alpha+\beta+N+s+1)}{(\alpha+\beta+2\,s)(\alpha+\beta+2\,s+2)(\alpha+\beta+2\,s-2)} & \text{if} \quad s \quad \text{even} \\ -2\,\frac{(\beta+s)(\alpha+\beta)(\alpha+\beta+2\,N+2)}{(\alpha+\beta+2\,s)(\alpha+\beta+2\,s+2)(\alpha+\beta+2\,s-2)} & \text{if} \quad s \quad \text{odd} \end{cases}. \tag{5.15}$$

We thus have a difference equation for the dual -1 Hahn polynomials in the form

$$U_2(s)\left(R_n^{(-1)}(y_{s+2}) - R_n^{(-1)}(y_s)\right) + U_1(s)\left(R_n^{(-1)}(y_{s+1}) - R_n^{(-1)}(y_s)\right) +$$
$$V_2(s)\left(R_n^{(-1)}(y_{s-2}) - R_n^{(-1)}(y_s)\right) + V_1(s)\left(R_n^{(-1)}(y_{s-1}) - R_n^{(-1)}(y_s)\right) = 2nR_n^{(-1)}(y_s),$$

where the coefficients $U_i(s), V_i(s)$ are provided in the above formulas.

## 6. Another form of the difference equation

The difference equation for the dual -1 Hahn polynomials can be presented in a more compact form if one notices that the grid $y_s$ satisfies the relations

$$y_{s\pm1} = \begin{cases} -y_s \mp 2 & \text{if} \quad s \quad \text{even} \\ -y_s \pm 2 & \text{if} \quad s \quad \text{odd} \end{cases}. \tag{6.1}$$

This property implies that the difference equation can be written as

$$E_1(x)\left(R_n^{(-1)}(x+4)-R_n^{(-1)}(x)\right)+E_2(x)\left(R_n^{(-1)}(x-4)-R_n^{(-1)}(x)\right)+$$

$$G_1(x)\left(R_n^{(-1)}(-x-2)-R_n^{(-1)}(x)\right)+G_2(x)\left(R_n^{(-1)}(-x+2)-R_n^{(-1)}(x)\right)=2nR_n^{(-1)}(x)$$

or, in operator form as

$$HR_n^{(-1)}(x)=2nR_n^{(-1)}(x). \tag{6.2}$$

The operator $H$ in (6.2) reads

$$H=E_1(x)T^4+E_2(x)T^{-4}+G_1(x)T^2R+G_2(x)T^{-2}R-(E_1(x)+E_2(x)+G_1(x)+G_2(x))I, \tag{6.3}$$

where the operators $T$ and $T^{-1}$ are the standard shift operators: $T^jf(x)=f(x+j)$, $j=0,\pm1,\pm2,\dots$, and $R$ is the reflection operator $Rf(x)=f(-x)$. $I$ denotes the identity operator.

The functions $E_i(x),G_i(x),\ i=1,2$ are simple rational functions in $x$.

For even $N=2,4,\dots$ we have

$$E_1(x)=\frac{(x+3-\alpha+\beta)\,(x+3-\alpha-\beta)\,(x-1-2\,N+\alpha+\beta)}{4\,(x+1)\,(3+x)},$$

$$E_2(x)=-\frac{(\alpha-1+\beta+x)\,(\alpha-1-\beta+x)\,(x-1+2\,N-\alpha-\beta)}{4\,(x-1)\,(x-3)},$$

$$G_1(x)=\frac{\left(\alpha^2-\beta^2\right)(x+\alpha+\beta-2\,N-1)}{(x^2-1)\,(x+3)},$$

$$G_2(x)=\frac{(2N+2-\alpha-\beta)\left((x+\alpha-1)^2-\beta^2\right)}{(x^2-1)\,(x-3)}$$

For odd $N=1,3,5,\dots$

$$E_1(x)=\frac{(x+\alpha+\beta+3)\,(x+\alpha-\beta+1)\,(x-\alpha-\beta-2\,N+1)}{4\,(x+1)\,(x+3)}$$

$$E_2(x)=-\frac{(x-\alpha-\beta-1)\,(x+\beta-\alpha-3)\,(x+\alpha+\beta+2\,N+1)}{4\,(x-1)\,(x-3)}$$

$$G_1(x)=\frac{(\alpha+\beta)\,(\alpha+\beta+2+2\,N)\,(x+1+\alpha-\beta)}{(1-x^2)\,(x+3)}$$

$$G_2(x)=\frac{(\alpha-\beta)\,(x-\alpha-\beta-1)\,(x+\alpha+\beta+2\,N+1)}{(1-x^2)\,(x-3)}$$

The form (6.2) of the difference equation is preferable because we here have the operator $H$ acting directly on the argument of the polynomials. The operator $H$ belongs to the class of Dunkl shift operators: it contains both simple shifts $T^j$ and the reflection operator $R$. Moreover, the operator $H$ preserves the space of polynomials: it transforms any polynomials of degree $n$ into a polynomial of the same degree $n$. Operators of this kind were considered in the theory of Bannai-Ito polynomials [11]. However, in contrast to the Bannai-Ito situation, the operator $H$ in the present case is of *second order*, while the Bannai-Ito polynomials are eigenfunctions of a Dunkl-shift operator of the first order [11]. Equivalently, this means that for generic values of the parameters $\alpha,\beta$, the dual -1 Hahn polynomials are eigenvectors of a 5-diagonal matrix (while the Bannai-Ito polynomials are eigenvectors of a 3-diagonal matrix).

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